Linearized Vortex Flows

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Steady vortex flows driven by a radial convection of angular momentum are considered. The incompressible Navier-Stokes equations are linearized by considering perturbations about both simple, nonrotating flows and strongly rotating flows, i.e., the equations are expanded for large and small Rossby numbers. Axial variations in a swirl superimposed upon a stagnation-point flow (with radial inflow) are considered for large Rossby numbers. An analytic solution is found for the axial decay of a given swirl. By considering flows for small Rossby numbers, it is found that, in flows dominated by rotation, the fluid motion is forced to be two-dimensional except in thin shear regions where necessary adjustments imposed by boundary conditions are made. The properties of these different shear layers are dependent on the gradient of the basic circulation of the flow as well as the Reynolds number.

Nomenclature

a = inflow gradient for the sink flow
f = radial variation of the dimensionless stream function
g = radial variation of the dimensionless circulation
h = axial variation of the dimensionless circulation
l = characteristic axial dimension of the problem
L = axial length of the container
n = number of the term in one of the series expansions
N = radial Reynolds number, Q/ν
p = pressure
Q = radial volume flow per unit axial length divided by 2π
r = radius
r_0 = characteristic radial dimension of the problem
Re_t = tangential Reynolds number, Γ_∞/ν
R_0 = Rossby number based on the radial flow, $Ql/\Gamma_\infty r_0$
u = radial velocity
v = tangential velocity
w = axial velocity
x = boundary-layer coordinate
z = axial coordinate
α = ratio of characteristic lengths squared $(r_0/l)^2$
β = coefficient in the expansion of Eq. (50)
Γ = dimensionless circulation, vr/Γ_∞
Γ_0 = principal part of the circulation distribution which depends on r only
Γ_1 = perturbation circulation
Γ_∞ = characteristic value of the circulation
δ = boundary-layer thickness
η = dimensionless radial coordinate, $(r/r_0)^2$
ν = kinematic viscosity
ξ = dimensionless axial coordinate, z/l
ρ = density
σ = coefficient in the expansion of Eq. (50)
$\hat{\psi}$ = stream function as defined in Eq. (5)
ψ = dimensionless stream function
ψ_1 = a perturbation to the stream function
Ω = rotation of the container

1. Introduction

AN interest in vortex flows has existed for many years. Studies concerning meteorology, the Ranque-Hilsch tube, the cyclone separator, wing theory, compressors, and others have each yielded considerable literature on vortex flows as related to that particular application. Recently, interest in confined vortices has been generated by certain concepts of advanced nuclear rocket propulsion using gaseous-core nuclear reactors[1] and of electrical power generation[2] using magnetohydrodynamic effects.

The present study primarily considers the type of flow shown in Fig. 1, where the fluid enters the region of interest tangentially, spirals radially inward, and exits axially at some smaller radius. The most common example of this type flow is the "bathtub" vortex. Since in such flows the radial and tangential velocities are required to be zero on the axis of symmetry, viscosity will generally play an important part in determining the flow, at least in some neighborhood of the axis. All of the present study is concerned with viscous flow.

The purpose of this study is to increase the understanding of vortex flows driven by a radial convection of angular momentum. It is well known that solving the full Navier-Stokes equations is a formidable problem, which can only be accomplished exactly in certain special cases. A useful method of extending these special solutions is to assume a perturbation about a known solution and to solve the resulting linearized equations. This method is used herein by linearizing about both nonrotating and strongly rotating flows.

The influence of rotation on a fluid motion may be indicated by the Rossby number, which represents the ratio of linear momentum to angular momentum. When the Rossby number is large, the influence of rotation is small, and it is possible to consider perturbations about any of the simple, known solutions of the axially symmetric Navier-Stokes equations. Görtler[3] solved the problem of the axial decay of a weak swirl in an axially symmetric jet by considering Schlichting's jet solution[4] with a weak swirl imposed. Talbot[5] as well as Collatz and Görtler[6] considered the problem of pipe flow with a weak swirl by linearizing the equations about the parabolic velocity distribution of Hagen-Poiseuille flow. Newman[7] studied a swirl perturbation about uniform axial flow. A perturbation about stagnation-point flow with radial inflow (a simple flow directly related to Fig. 1) is considered in the present study.

In many real flows, the rotational velocity is the dominant feature of the flow, in which case some information as to the nature of the flow can be obtained by considering a perturbation for small Rossby numbers about a specified circulation distribution.

Perturbation flows in a fluid rotating with a uniform angular velocity have been considered by a number of investigators beginning with Taylor, Proudman, and Grace during the period 1915–1925. Their work, as well as the more recent work of Stewartson, Morgan, Frankael, and Long, was surveyed by Squire.[8] The work reviewed by Squire was mostly related to the inviscid motion of bodies in a rotating fluid.

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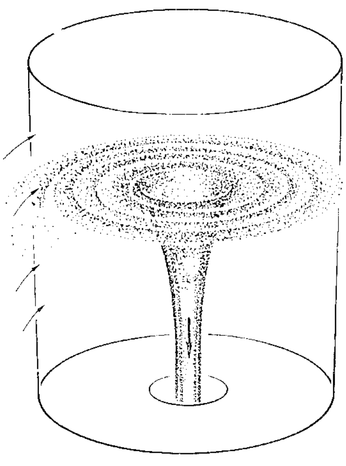

Fig. 1 Vortex flow in which fluid enters tangentially, spirals radially inward, and exits axially at some smaller radius.

The studies of Proudman[9] and Stewartson[10] are of more particular interest to the present investigation. Proudman considered the viscous flow between two concentric spheres rotating about the same axis with almost equal angular velocities. Stewartson considered the similar problem of the flow between two coaxial rotating disks, each having a small arbitrary angular velocity superimposed on a constant angular velocity. Because of the somewhat simpler geometry, Stewartson's solution was more nearly complete than that given by Proudman. When the tangential Reynolds number is large, the departure of the flow from that of solid-body rotation was found to occur in thin shear regions. The thicknesses of these shear regions were $O(Re_t^{-1/2})$, $O(Re_t^{-1/3})$, or $O(Re_t^{-1/4})$, depending upon the function of the particular region. The corresponding unsteady motion in these shear layers has been considered by Greenspan and Howard.[11]

The linearization for small Rossby numbers to be considered here differs from that previously considered in the literature principally in two ways: specified circulation distribution other than that of solid-body rotation is considered; also, a specific example directly related to the flow of Fig. 1 is solved.

II. Fundamental Equations

The equations of steady motion for an incompressible fluid with constant viscosity in cylindrical coordinates assuming axial symmetry are

$$[\partial(ru)/\partial r] + [\partial(rw)/\partial z] = 0 \tag{1}$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2}$$

$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{3}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right) \tag{4}$$

where r is the radial coordinate; z the axial coordinate; u, v, and w the radial, tangential, and axial components of velocity, respectively; p the pressure; ρ the density; and ν the kinematic viscosity.

For the general problem, the number of dependent variables may be reduced by defining the usual axisymmetric stream function

$$u = (1/r)(\partial\hat{\psi}/\partial z) \qquad w = -(1/r)(\partial\hat{\psi}/\partial r) \tag{5}$$

and by eliminating the pressure by cross differentiation of Eqs. (2) and (4). The resulting equations can be somewhat simplified by writing them in terms of r^2. For convenience, the following dimensionless parameters are also introduced:

$$\eta = r^2/r_0^2 \qquad \Gamma = vr/\Gamma_\infty$$
$$\xi = z/l \qquad \psi = \hat{\psi}/Ql \tag{6}$$

where r_0, l, Γ_∞, and Q are characteristic dimensional quantities which must be chosen separately for each problem considered. For the flow sketched in Fig. 1, a suitable r_0 may be the radius of the exhaust hole, l the length of the vortex chamber, $2\pi\Gamma_\infty$ the circulation of the entering flow, and $2\pi Q$ the volume flow per unit length. The resulting equations are

$$\frac{\partial\psi}{\partial\xi}\frac{\partial\Gamma}{\partial\eta} - \frac{\partial\psi}{\partial\eta}\frac{\partial\Gamma}{\partial\xi} = \frac{2\eta}{N}\frac{\partial^2\Gamma}{\partial\eta^2} + \frac{\alpha}{2N}\frac{\partial^2\Gamma}{\partial\xi^2} \tag{7}$$

and

$$\Gamma\frac{\partial\Gamma}{\partial\xi} = R_0^2\left\{4\eta^2\left[\frac{\partial\psi}{\partial\xi}\frac{\partial^3\psi}{\partial\eta^3} - \frac{\partial\psi}{\partial\eta}\frac{\partial^3\psi}{\partial\xi\partial\eta^2} - \frac{2}{N}\left(2\frac{\partial^3\psi}{\partial\eta^3} + \eta\frac{\partial^4\psi}{\partial\eta^4}\right)\right] + \alpha\left[\eta\frac{\partial\psi}{\partial\xi}\frac{\partial^3\psi}{\partial\eta\partial\xi^2} - \frac{\partial\psi}{\partial\xi}\frac{\partial^2\psi}{\partial\xi^2} - \eta\frac{\partial\psi}{\partial\xi}\frac{\partial^3\psi}{\partial\xi^3} - \frac{1}{N}\left(4\eta^2\frac{\partial^4\psi}{\partial\eta^2\partial\xi^2} + \frac{\alpha\eta}{2}\frac{\partial^4\psi}{\partial\xi^4}\right)\right]\right\} \tag{8}$$

As seen from Eqs. (7) and (8) the flow is governed by three dimensionless parameters: $N = Q/\nu$, a Reynolds number; $\alpha = (r_0/l)^2$, the ratio of characteristic lengths squared; and $R_0 = Ql/\Gamma_\infty r_0$, the ratio of volume flow to circulation times the radius, commonly called the Rossby number in meteorological literature.

The system of Eqs. (7) and (8), of course, still contains the full complexity of the original axially symmetric Navier-Stokes equations. Consequently, exact or "nearly exact" solutions can be expected to be found in only rather special cases. It is the purpose of this study to investigate the linearization of these equations for flows related to that of Fig. 1. A more general consideration of Eqs. (7) and (8) for flows related to Fig. 1 is given in Ref. 12.

III. Perturbations about Flows with No Swirl

Any of the simple, known solutions of the axially symmetric Navier-Stokes equations may be extended to include flows with weak swirl. When the Rossby number is large, the effect of circulation on the stream function is small. The stream function may then be taken as given, up to a small perturbation, by the known solution of a nonrotating problem. Examples of this approach have been given in the introduction.

A simple flow directly related to Fig. 1 is that of axisymmetric stagnation-point flow (with radial inflow). For this flow, both the outer radius and the exhaust radius shown in Fig. 1 become infinite and the plane of $z = 0$ becomes a plane of symmetry. Let us consider a circulation superimposed upon such a stagnation-point inflow and assume that this circulation has a small effect upon the given stream function. Burgers[13] and Rott[14] have considered the special case of a tangential velocity, which is a function only of the radius, superimposed upon stagnation-point inflow. For this special case, the stream function is independent of the circulation. Thus, the linearization assumed here implies either that the circulation is weak (large R_0) or that the circulation is almost independent of z.

Let

$$\psi(\eta, \xi) = \eta\xi + \psi_1(\eta, \xi) \tag{9}$$

with $\psi_1 \ll 1$. The perturbation equations of motion can then be written as

$$\eta\frac{\partial\Gamma}{\partial\eta} - \xi\frac{\partial\Gamma}{\partial\xi} = \frac{2\eta}{N}\frac{\partial^2\Gamma}{\partial\eta^2} + \frac{\alpha}{2N}\frac{\partial^2\Gamma}{\partial\xi^2} + O(\psi_1\Gamma) \tag{10}$$

and

$$\frac{\Gamma}{R_0^2}\frac{\partial\Gamma}{\partial\xi} = 4\eta^2\left[\eta\frac{\partial^3\psi_1}{\partial\eta^3} - \xi\frac{\partial^3\psi_1}{\partial\xi\partial\eta^2} - \frac{2}{N}\left(2\frac{\partial^3\psi_1}{\partial\eta^3} + \eta\frac{\partial^4\psi_1}{\partial\eta^4}\right)\right] + \alpha\left[\eta^2\frac{\partial^3\psi_1}{\partial\eta\partial\xi^2} - \eta\frac{\partial^2\psi_1}{\partial\xi^2} - \eta\xi\frac{\partial^3\psi_1}{\partial\xi^3} - \frac{1}{N}\left(4\eta^2\frac{\partial^4\psi_1}{\partial\eta^2\partial\xi^2} + \frac{\alpha\eta}{2}\frac{\partial^4\psi_1}{\partial\xi^4}\right)\right] + O(\psi_1^2) \quad (11)$$

It is clear from Eq. (11) that, to meet the restriction that $\psi_1 \ll 1$, it is necessary to have $R_0 \gg 1$ if the circulation varies with the axial coordinate. This is the requirement of weak swirl. The characteristic quantities to be used in defining the parameters N, R_0, and α will be discussed later.

Equation (10) can be solved by separating the variables. Assume

$$\Gamma = g(\eta)h(\xi) \quad (12)$$

The resulting two equations to be solved are

$$(\alpha/2N)h'' + \xi h' - kh = 0 \quad (13)$$

$$(2/N)\eta g'' - \eta g' + kg = 0 \quad (14)$$

where k is the separation constant. To determine the axial decay of the circulation, let us consider $\Gamma(\eta, \xi_0)$ as given and initially assume that no further circulation is introduced into the flow for $\xi > \xi_0$. The boundary conditions on Γ then are

$$\Gamma(\infty, \xi) = \Gamma(0, \xi) = 0 \qquad \Gamma(\eta, \xi \to \infty) \to 0$$

$$\Gamma(\eta, \xi_0) = F(\eta) \quad (15)$$

The eigenvalue problem then is to look for values of k that permit solutions of Eq. (14) satisfying the conditions

$$g(0) = g(\infty) = 0 \quad (16)$$

The general solution of Eq. (14) for arbitrary k can be given in terms of confluent hypergeometric functions. But a much simpler solution satisfying Eq. (16) is possible (providing $N < 0$ corresponding to radial inflow) when $k = -n$, with n a positive integer. In this case

$$g_n = (d^{n-1}/d\eta^{n-1})(\eta^n e^{N\eta/2}) \quad (17)$$

For $k = -n$, Eq. (13) can be transformed into Weber's equation. Then the solution that is finite for $\xi \geq 0$ and goes to zero as $\xi \to \infty$ is

$$h_n = e^{-(N/2\alpha)\xi^2} D_{-n}\{[-(2N/\alpha)]^{1/2}\xi\} \quad (18)$$

with $D_n(x)$ equal to the Weber function.[15] As $\xi \to \infty$

$$h_n \to \xi^{-n} \quad (19)$$

To show that the general solution of Eq. (10) satisfying the conditions of Eq. (15) can be represented in the form

$$\Gamma(\eta, \xi) = \sum_{n=1}^{\infty} c_n g_n(\eta) h_n(\xi) \quad (20)$$

one only need show that the g_n's of Eq. (17) form a complete set. It can be seen from Eq. (17) that $g_n \exp(-N\eta/2)$ is a polynomial of degree n. Thus, any function F analytic at $\eta = 0$ can be expanded in terms of the functions $g_n \exp(-N\eta/2)$. The resulting expansion for F is valid for $\eta < \eta_0$, the smallest singular point of F. Since the only singularity of $\exp(N\eta/2)$ occurs at $\eta = \infty$, it follows that F can be expanded in terms of g_n also for $\eta < \eta_0$.† Thus, Eq. (20) represents the general solution satisfying the conditions of Eq. (15) as long as the given Γ distribution at $\xi = \xi_0$ has no singularities between $\eta = 0$ and $\eta = \infty$.

An interesting point about the general solution Eq. (20) is that the higher-numbered eigenfunctions decay more rapidly with ξ, so that as $\xi \to \infty$

$$\Gamma \to (c/\xi)\eta e^{N\eta/2} \quad (21)$$

regardless of the specified distribution at $\xi = \xi_0$.

Because of the linearity of the system, a Burgers-Rott solution can also be added to the solution given in Eq. (20); in fact, it can be considered as the zeroth eigenfunction in Eq. (20). Their solution for Γ can be obtained by setting $h_0 =$ const and $k = 0$ in Eqs. (13) and (14) with the boundary conditions that $\Gamma(0, \xi) = 0$ and $\Gamma(\infty, \xi) = A$. The resulting solution is

$$g_0 = A(1 - e^{N\eta/2}) \quad (22)$$

The solution has been obtained for the axial decay of circulation for stagnation-point inflow when the swirl either is weak or is almost uniform with z. The three lowest radial eigenfunctions (g_0, g_1, and g_2) are plotted in Fig. 2 for $N = -2$.

The characteristic quantities Q, Γ_∞, r_0, and l, used in the definitions of N, R_0, and α have not been fixed for this problem. Since there is no geometrical radius for this problem, and since the radius appears multiplied by the radial Reynolds number in the solutions, it is appropriate to base r_0 on the kinematic viscosity divided by a velocity. Then it is consistent to have

$$Q = -ar_0^2 \qquad r_0 = (2\nu/a)^{1/2} \qquad N = -2 \quad (23)$$

The r_0 defined in this manner is the radius of the viscous core as discussed by Rott.[14] The circulation Γ_∞ can be chosen as the asymptotic value that vr approaches as $r \to \infty$ so that $A = 1$. (If vr approaches zero as $r \to \infty$, then Γ_∞ might well be chosen as the maximum value of vr at $\xi = \xi_0$.) Two cases must be considered in choosing l, depending upon whether z_0, the position at which the radial distribution of Γ is prescribed, is finite or zero. If z_0 is finite, then it is a geometrical length and can appropriately be used for l. But if $z_0 = 0$, then l must also be based on the viscosity so that it is appropriate to take $l = r_0$. In the latter case, $\alpha = 1$ and $R_0 = 2\nu/\Gamma_\infty$ so that $R_0 \gg 1$ imposes a strong restriction on the circulation. In the former case, R_0 increases with z_0 so that asymptotically R_0 always becomes large for any Γ_∞.

The effect of the swirl on the stream function will be considered only for a specific case. First, consider Γ given by the

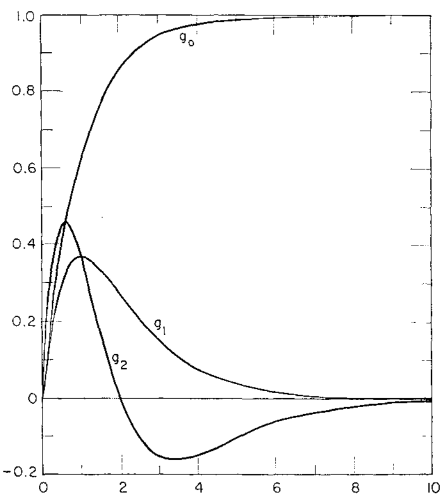

Fig. 2 The three leading eigenfunctions in the expansion of the circulation in the problem of stagnaton-point inflow with a weakly decaying swirl.

† The g_n's are in fact related to the associated Laguerre polynomials[16]

$$L_{n-1}{}^1(x) = \frac{(n-1)!}{n!}\frac{e^x}{x}\frac{d^{n-1}}{dx^{n-1}}x^n e^{-x}$$

which are a complete set.

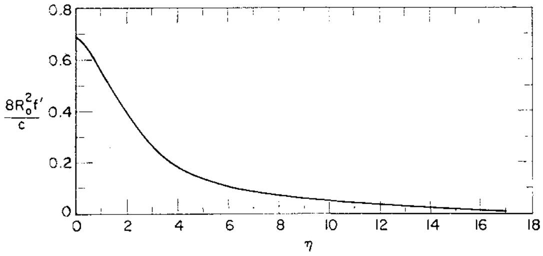

Fig. 3 Radial distribution of perturbation axial velocity caused by a weakly decaying swirl superimposed upon stagnation-point inflow.

two leading terms of its asymptotic form for large ξ, with $A = 1$,

$$\Gamma = 1 - e^{-\eta} + (c\eta/\xi)e^{-\eta} \tag{24}$$

Second, let $\alpha = 0$ in Eq. (11). This corresponds to looking at the asymptotic form of the equation for large ξ. Note, that if $\alpha = 0$ in Eq. (13), then the h_n's are given simply by $h_n = \xi^{-n}$. Under these conditions, Eq. (11) becomes

$$-\eta \frac{\partial^4\psi_1}{\partial\eta^4} - 2\frac{\partial^3\psi_1}{\partial\eta^3} + \xi\frac{\partial^3\psi_1}{\partial\xi\partial\eta^2} - \eta\frac{\partial^3\psi_1}{\partial\eta^3} = \frac{ce^{-\eta}}{4R_0^2\eta\xi^2} \times \left[1 - e^{-\eta}\left(1 - \frac{c\eta}{\xi}\right)\right] \tag{25}$$

From the form of Eq. (25) it is clear that ψ_1 can be expanded in inverse powers of ξ. Assuming no other disturbance to the stream function other than the decay of circulation and keeping only the leading term in the expansion, one can represent ψ_1 as

$$\psi_1 = f(\eta)/\xi^2 + \ldots \tag{26}$$

A solution for f satisfying the conditions that f' and all its higher derivatives approach zero as $\eta \to \infty$ is

$$f' = \frac{c}{8R_0^2}\left[e^{-\eta}\int_0^\eta \frac{e^\eta - e^{-\eta}}{\eta}\,d\eta + 2\int_\eta^\infty \frac{e^{-\eta}(1 - e^{-\eta})}{\eta} + C_1e^{-\eta}\right] \tag{27}$$

This function is plotted in Fig. 3 for $C_1 = 0$. It shows that if c is positive, so that the absolute magnitude of the circulation is decaying as ξ increases, then the axial velocity $\partial\psi/\partial\eta$ will be increased at a slower rate on the axis than it is in an annulus about the axis. This is a result of the reduced axial pressure gradient on the axis. In strongly rotating flows, this is accentuated and the axial pressure gradient can be reversed to give the reversed flow cell often encountered.[12]

In summary, the eigenfunctions for the axial decay of circulation for stagnation-point inflow have been obtained for the case in which the effect of tangential velocity is restricted to a small perturbation of the stream function. To meet this restriction it is necessary to have

$$1/R_0^2|\partial\Gamma/\partial\xi| \ll 1 \tag{28}$$

That is, either the Rossby number is large or Γ must be very nearly independent of z. Inversely a very small variation of Γ with z can have a large effect on ψ if the Rossby number is small. This case is discussed further in the next section.

IV. Perturbations About Rotating Flows

Some information about flows dominated by rotation can be obtained by considering perturbations about a specified Γ_0 distribution. Small perturbation flows about rotating flows where the basic rotation distribution is other than that of uniform rotation are rather difficult to realize in a laboratory experiment. However, since the investigation of such flows should aid in explaining some of the features that occur in real rotating flows, the flow here will be linearized about a given arbitrary rotation distribution. Since the Rossby number of motion dominated by rotation is small, it can be seen from Eq. (8) that the given Γ_0 distribution should be restricted to a function of the radial coordinate only. Therefore, let

$$\Gamma = \Gamma_0(\eta) + \Gamma_1(\eta, \xi) \tag{29}$$

and assume that

$$\Gamma_1 \ll 1 \tag{30}$$

If a straight Rossby number expansion is used, with α and N considered as order-one quantities, as was carried out by Lewellen,[12, 17] then from Eqs. (7) and (8) it follows that

$$\Gamma_0(\partial\Gamma_1/\partial\xi) = O(R_0^2) \tag{31}$$

$$\Gamma_0(\partial\psi/\partial\xi) = (2\eta/N)\Gamma_0'' \tag{32}$$

therefore,

$$\psi = \xi f_1(\eta) + f_0(\eta) \tag{33}$$

with f_1 and Γ_0 related by the equation

$$2\eta\Gamma_0'' - Nf_1\Gamma_0' = 0 \tag{34}$$

Such a flow must be two-dimensional with the radial and tangential velocities depending only on r and the axial velocity linear with z.

If the boundary conditions are such that the flow cannot be two-dimensional everywhere, then some of the higher-derivative shear terms must be retained in Eqs. (31) and (32) to permit the boundary conditions to be satisfied. The linearized equations of motion including the shear terms then are

$$\Gamma_0'\frac{\partial\psi}{\partial\xi} = \frac{1}{N}\left(2\eta\frac{\partial^2\Gamma_1}{\partial\eta^2} + \frac{\alpha}{2}\frac{\partial^2\Gamma_1}{\partial\xi^2} + 2\eta\Gamma_0''\right) \tag{35}$$

$$\Gamma_0\frac{\partial\Gamma_1}{\partial\xi} = -\frac{R_0^2}{N}\left[8\eta^2\left(2\frac{\partial^3\psi}{\partial\eta^3} + \eta\frac{\partial^4\psi}{\partial\eta^4}\right) + \alpha\eta\left(4\eta\frac{\partial^4\psi}{\partial\eta^2\partial\xi^2} + \frac{\alpha}{2}\frac{\partial^4\psi}{\partial\xi^4}\right)\right] \tag{36}$$

Equations (35) and (36) will not be solved for the given boundary conditions of a particular physical flow; rather, the different shear layers occurring in strongly rotating flows will be examined separately. A physical problem can then be solved by appropriately matching these shear layers to fulfill given boundary conditions. Here, a shear layer is defined as a thin region in which the gradients of a variable are much larger than the variable itself. The different order shear layers can be obtained from Eqs. (35) and (36) by making three different boundary-layer approximations.

In a shear layer of thin axial extent (type-I boundary layer), the characteristic axial dimension is the layer thickness δ_1. Thus, $\alpha = (r_0/\delta_1)^2 \gg 1$, and the equations reduce to

$$\Gamma_0'\frac{\partial\psi}{\partial\xi} = \frac{\alpha}{2N}\frac{\partial^2\Gamma_1}{\partial\xi^2} + \frac{2\eta}{N}\Gamma_0'' \tag{37}$$

$$\Gamma_0\frac{\partial\Gamma_1}{\partial\xi} = -\frac{R_0^2\alpha^2}{2N}\eta\frac{\partial^4\psi}{\partial\xi^4} \tag{38}$$

Note that in these two equations for Γ_1 and ψ, differentiation occurs only with respect to ξ, so that η can be regarded as a parameter rather than a coordinate. Substituting Eq. (37) into Eq. (38) gives

$$\Gamma_0\Gamma_0'\frac{\partial\Gamma_1}{\partial\xi} = -\frac{R_0^2\alpha^3}{4N^2}\eta\frac{\partial^5\Gamma_1}{\partial\xi^5} \tag{39}$$

Therefore, for consistency, $R_0^2\alpha^3/N^2 = O(1)$, so that

$$\frac{\delta_1}{r^0} = O\left(\frac{\nu}{\Gamma_\infty}\right)^{1/2} = O\left(\frac{1}{\mathrm{Re}_t^{1/2}}\right) \tag{40}$$

The axial boundary layer thus has the same order of thickness as that usually found in boundary-layer theory.

In a shear layer of thin radial extent located at a radius r_0

$$\eta = 1 + \delta x \tag{41}$$

and Eqs. (35) and (36) reduce to

$$\Gamma_0' \frac{\partial \psi}{\partial \xi} = \frac{2}{N\delta^2} \frac{\partial^2 \Gamma_1}{\partial x^2} + \frac{2}{N} \Gamma_0'' \tag{42}$$

$$\Gamma_0 \frac{\partial \Gamma_1}{\partial \xi} = -\frac{8R_0^2}{N\delta^4} \frac{\partial^4 \psi}{\partial x^4} \tag{43}$$

From Eqs. (42) and (43) it is possible to distinguish two types of radial shear layers. Eliminating ψ from Eqs. (42) and (43) gives

$$\Gamma_0 \Gamma_0' \frac{\partial^2 \Gamma_1}{\partial \xi^2} = -\frac{16R_0^2}{N^2\delta^6} \frac{\partial^6 \Gamma_1}{\partial x^6} \tag{44}$$

and setting $R_0^2/N^2\delta^6 = O(1)$ gives

$$\delta_2 = O[Re_t^{-1/3}(l/r_0)^{1/3}] \tag{45}$$

in what will be called a type-II boundary layer.

The type-III shear layer occurs when $R_0^2\psi/N\delta^4\Gamma_1 \ll 1$. In this case, from Eq. (43),

$$\Gamma_1 = \Gamma_1(x) \tag{46}$$

and

$$\delta_3 = O[N(\psi/\Gamma_1)]^{-1/2} \tag{47}$$

This is the layer that may be $O(Re_t^{-1/4})$ as will be seen later.

Each of these three shear layers will be considered in some detail. Then an example will be given of patching these flows together to get the complete flow in a rotating container.

In the case of the type-I axial boundary layer, if it is assumed that the boundary conditions are that $\Gamma_1 = 0$ at $\xi = 0$ and approaches asymptotically a finite value as $\xi \to \infty$, then the solution to Eq. (39) is

$$\Gamma_1 = g(\eta)[1 - e^{-\bar{\xi}} \cos\bar{\xi}] \tag{48}$$

with $g(\eta)$ arbitrary and $\bar{\xi} = (\Gamma_0\Gamma_0'/\eta)^{1/4} Re_t^{1/2}(z/r_0)$ provided that $\Gamma_0\Gamma_0' > 0$. If $\Gamma_0\Gamma_0' \leq 0$, it is impossible to satisfy the desired boundary conditions. Thus, no solution is possible for Γ_0 decreasing with increasing r. From Eq. (37) the solution for ψ, satisfying the same boundary conditions that $\psi = 0$ at $\xi = 0$ and approaches a finite value as $\xi \to \infty$, is found to be

$$\psi = -\frac{Re_t}{2N\Gamma_0'} g(\eta) \left(\frac{\Gamma_0\Gamma_0'}{\eta}\right)^{1/4} \times \left[1 - (\sin\bar{\xi} + \cos\bar{\xi})e^{-\bar{\xi}} + O\left(\frac{1}{Re_t}\right)\right] \tag{49}$$

For the special case of $\Gamma_0 = \eta$ (rigid rotation) and $g(\eta) = 1$, the solution given in Eqs. (48) and (49) reduces to the Ekman spiral.[18] There are two notes of special interest about the more general spiral represented by Eqs. (48) and (49). The first is the fact that the solutions remain valid when multiplied by a general function of the radius $g(\eta)$, provided only that it is the same function for both the circulation and the stream function. The second is the way in which the boundary-layer thickness varies as the main Γ_0 distribution departs from that of rigid rotation. If the basic circulation distribution grows with radius faster than $\Gamma_0 = \eta$, then the over-all boundary-layer thickness is decreased and becomes a function of the radius, decreasing as the radius increases. On the other hand, if the basic circulation distribution grows with radius more slowly than rigid rotation, then the boundary-layer thickness is increased over all and also increases with increasing radius. Equations (48) and (49) show that the boundary layer becomes infinite as Γ_0 approaches the constant value of potential flow. As stated previously, it is impossible to obtain this type of boundary-layer solution for Γ_0 decreasing with increasing radius.

King and Lewellen[19] have found similarity solutions for flow in a boundary layer of the type considered here for the nonlinear problem with boundary conditions that permit a radial power-law variation of circulation. The numerical solutions obtained for the nonlinear problem exhibited oscillations and a boundary-layer character similar to that of the linear solution given here. In the nonlinear case, there was no way of investigating the nature of the singularity in the boundary-layer thickness for the potential vortex. In the linear case here, the boundary-layer thickness grows as $(\Gamma_0')^{-1/4}$.

In the type-II radial boundary layer, where the flow is defined by Eq. (44), it is possible to take $\Gamma_0\Gamma_0'$ as constant and equal to its value at $r_0(\eta = 1)$. The solution thus can be given as

$$\Gamma_1 = \sum_n \sum_{j=1}^{3} [a_{nj} \exp(\sigma_{nj}x + i\beta_n\xi) + b_{nj} \exp(\sigma_{nj}x - i\beta_n\xi)] \tag{50}$$

with the σ_{nj}'s the three roots of

$$\sigma_n^3 = (N^2\delta_2^6/16R_0^2)\Gamma_0\Gamma_0'\beta_n \tag{51}$$

The β_n's are eigenvalues to be determined by the boundary conditions. This boundary-layer solution was given by Stewartson[10] for rigid rotation. This layer is also seen to become infinite as Γ_0 becomes a constant. In the example to be given, it will be seen that this type-II boundary layer is needed only when the boundary conditions cannot be satisfied by the other two.

In the type-III shear layer that occurs when $R_0^2\psi/N\delta^4\Gamma_1 \ll 1$, Eqs. (42) and (43) reduce to

$$\Gamma_0' \frac{\partial \psi}{\partial \xi} = \frac{2}{N\delta_3^2} \frac{\partial^2 \Gamma_1}{\partial x^2} + \frac{2}{N} \Gamma_0'' \tag{52}$$

$$\partial \Gamma_1/\partial \xi = 0 \tag{53}$$

Therefore,

$$\Gamma_1 = \Gamma_1(x) \qquad \psi = \xi f_1(x) + f_0(x) \tag{54}$$

and

$$2\Gamma_1'' - N\delta_3^2 f_1 \Gamma_0' = -2\Gamma_0'' \tag{55}$$

The forms of f_1 and f_0 are arbitrary as far as the equations in this layer are concerned; they are left completely free to be determined by the boundary conditions on the layer. Since Γ_1 and Γ_0 are both functions only of the radius, this distinction between the basic rotation and the perturbation rotation is not necessary in this case. The type-III shear layer can be considered with the main flow; i.e., Eq. (55) can be included with Eq. (34) and written simply as

$$2\eta\Gamma'' - Nf_1\Gamma' = 0 \tag{56}$$

In this way the full Γ distribution may be determined as a function of a specified stream function. This way of considering the flow solution in terms of an expansion for large ratios of circulation to stream function has been discussed by Lewellen.[12, 17]

V. Flow through a Rotating Container

The role of these different shear layers probably can be best illustrated by means of a specific flow example. For this purpose, consider the flow in a rotating chamber with porous cylindrical walls, for the case in which the speed of rotation of the chamber is much larger than the flow velocity through it. This is the flow of Fig. 1 bounded by porous cylinders at r_0 and r_i and solid disks at $z = \pm L/2$ when all of the bound-

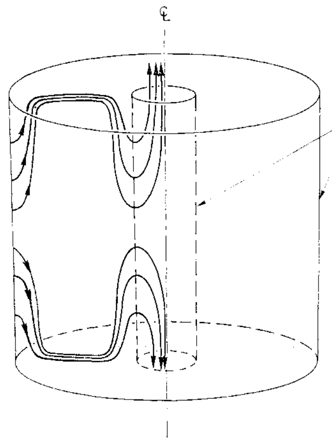

Fig. 4 Rotating cylindrical container showing paths of radial flow through it.

aries rotate uniformly. Also assume that the Reynolds number of rotation is large so that the principal axial variation of the velocities can be expected to occur in type-I boundary layers on the end walls of the chamber. The flow picture anticipated is shown in Fig. 4. The radial flow, although uniformly distributed over the length of the chamber at both cylindrical side walls, goes to the type-I boundary layers on the axial end walls. This redistribution of the radial flow occurs within a type-III boundary layer on the cylindrical walls.

Within the boundary layers on the end walls, Eqs. (37) and (38) apply with $\Gamma_0 = \eta$, and the solution for the perturbation circulation and stream function from Eqs. (48) and (49) is

$$\Gamma_{1bl} = g(\eta)(1 - e^{-\bar{\xi}}\cos\bar{\xi}) \tag{57}$$

$$\psi_{bl} = -g(\eta)(\mathrm{Re}_t/2N)[1 - e^{-\bar{\xi}}(\cos\bar{\xi} + \sin\bar{\xi})] \tag{58}$$

with

$$\bar{\xi} = [(L/2 - |z|)/\delta_1] \qquad \delta_1 = r_0/\mathrm{Re}_t^{1/2} \tag{59}$$

with the coordinate system chosen so that the plane $z = 0$ bisects the chamber. Of course, it must be kept in mind that in Eqs. (58) and (59) Γ and ψ are dimensionless and, to be made dimensional, should be multiplied by $\Gamma_\infty = \Omega r_0^2$ (with Ω the rotation of the container) and $Q\delta_1$ (with $2\pi Q$ the radial volume flow per unit length), respectively.

Outside the two boundary layers on the top and bottom walls of the chamber, there is nothing to destroy the two-dimensionality of the motion, and Eq. (56) may be applied. With the nonlinear term $Nf_1\Gamma_1'$ neglected, the equation of motion in this region is

$$2\eta\Gamma_1'' - Nf_1 = 0 \tag{60}$$

The solution of the flow in the chamber is obtained by matching these regions at the edge of the top and bottom boundary layers. By continuity, the matching condition on the stream function is

$$f_1 + (2\delta_1/L)\psi_{bl}(\infty, \eta) = 1 \tag{61}$$

Applying Eq. (61) and the matching condition that Γ_1 be equal at the outer edge of the boundary layer to its value in the outer flow, one finds that Eq. (60) becomes

$$2\eta g'' - Re_t^{1/2}(r_0/L)g = N \tag{62}$$

If it is assumed that the porous cylindrical walls of the chamber are located at $r = r_0$ and at $r = r_i$, then g is zero at these two points. The general solution of Eq. (62) with these boundary conditions is

$$g = R_0(Re_t)^{1/2}y[AI_1(2y) + BK_1(2y)] \tag{63}$$

where I_1 and K_1 are the first-order Bessel functions with imaginary arguments, and

$$A = \frac{K_1(2y_i)/y_0 - K_1(2y_0)/y_i}{I_1(2y_0)K_1(2y_i) - I_1(2y_i)K_1(2y_0)}$$

$$B = \frac{I_1(2y_0)/y_i - I_1(2y_i)/y_0}{I_1(2y_0)K_1(2y_i) - I_1(2y_i)K_1(2y_0)}$$

$$y = (2r_0L)^{-1/2}(Re_t)^{1/4}r$$

If the tangential Reynolds number is sufficiently large, then y will be large everywhere in the interval $y_0 \geq y \geq y_i$ and the asymptotic expansion of I_1 and K_1 can be used, in which case

$$g = R_0(Re_t)^{1/2}\left\{y^{1/2}\left[\frac{e^{2(y-y_0)}}{y_0^{1/2}} + \frac{e^{2(y_i-y)}}{y_i^{1/2}}\right] - 1\right\} \tag{64}$$

Collecting the preceding equations, one sees that the complete solution for the circulation and stream function within the rotating chamber is

$$vr = \Omega r^2 + Q(L/r_0)(\Omega r_0^2/\nu)^{1/2}\bar{g}(r)(1 - e^{-\bar{\xi}}\cos\bar{\xi}) \tag{65}$$

$$\hat{\psi} = QL\{(1 + \bar{g})[(2|z|/L) - 1] - \bar{g}[1 - e^{-\bar{\xi}}(\cos\bar{\xi} + \sin\bar{\xi})]\} \tag{66}$$

with

$$\bar{g} = r^{1/2}\left\{\frac{1}{r_0^{1/2}}\exp\left[\left(\frac{2r_0}{L}\right)^{1/2}\left(\frac{\Omega r_0^2}{\nu}\right)^{1/4}\left(\frac{r}{r_0} - 1\right)\right] + \frac{1}{r_i^{1/2}}\exp\left[\left(\frac{2r_0}{L}\right)^{1/2}\left(\frac{\Omega r_0^2}{\nu}\right)^{1/4}\left(\frac{r_i}{r_0} - \frac{r}{r_0}\right)\right]\right\} - 1 \tag{67}$$

$$\bar{\xi} = [(L/2) - |z|](\Omega/\nu)^{1/2} \tag{68}$$

From Eq. (65) it is seen that outside of an axial boundary layer of $O(Re_t^{-1/2})$ on the end walls and a radial boundary layer of $O[(L/r_0)^{1/2}Re_t^{-1/4}]$ on the porous side walls, the perturbation circulation is a constant of $O(R_0Re_t^{1/2})$. If the flow QL through the rotating chamber is negative, i.e., radially inward, then the perturbation circulation adds to the basis rotation; if the flow is radially outward, then the perturbation circulation subtracts from the basic rotation. From Eqs. (66) and (67) it can be seen that the radial flow is confined to the boundary layers as anticipated in Fig. 4. Although the flow is uniformly distributed over the length of the chamber at both cylindrical side walls, it goes to the axial end walls in the radial boundary layer of $O[(L/r_0)^{1/2}Re_t^{-1/4}]$. The total flow is then confined to the axial boundary layer of $O(Re_t^{-1/2})$ for the traverse of the chamber until it reaches the radial boundary layer on the opposite side wall. Between the two radial boundary layers, the axial boundary layers on the two end walls reduce to classical Ekman boundary layers. Experiments, which check the perturbation circulation obtained in these layers, have been made by Faller,[20] who has also carried the theory in these regions to higher order in an R_0 expansion.

If the rotation Reynolds number is such that $Re_t^{-1/4} \times (L/r_0)^{1/2}$ is not a very small number but $Re_t^{-1/2}$ continues to be small, then the solution of Eqs. (65) and (66) remains valid, but Eq. (63) must be used to determine $\bar{g}$ rather than Eq. (64). For this case, the two radial boundary layers tend to merge and lose their identity.

In Refs. 21 and 22 the corresponding nonlinear problem of flow through a cylindrical container has been solved numerically. The outer porous cylinder is assumed to rotate in the same manner as for the linearized problem here, but the end walls are held stationary in the nonlinear problem. The analytic solution given here yields most of the features found in the nonlinear numerical solution.

The $O(Re_t^{-1/2})$ boundary layer on the end walls and the $O[(L/r_0)^{1/2}Re_t^{-1/4}]$ on the side walls are the same as that obtained by Stewartson[10] in his example of a rotating cylindrical chamber, in which the cylindrical walls rotate at a velocity slightly different from that of the end walls. The present example is somewhat simpler since the type-II shear layer represented by Eqs. (44) and (45) was not required. This added shear layer is required in Stewartson's example because the remaining two cannot satisfy the boundary conditions at the solid cylindrical walls.

To see how the third region joins the two in the present example, consider the flow through the outer cylindrical wall to be some sort of jet at the center rather than distributed uniformly across the length. Since the flow will remain symmetric about the plane $z = 0$, the solution for the perturbation circulation in this type-II layer can be given from Eq. (50) as

$$\Gamma_1 = \sum_n \sum_{j=1}^{3} a_{nj} \exp(-\sigma_{nj}x) \cos\beta_n\xi + C \tag{69}$$

with $\sigma_n^3 = \beta_n/16$, $\delta_2 = (l/r_0)^{1/3}Re_t^{-1/3}$, and $x = (1 - r^2/r_0^2)/\delta_2$. From Eq. (42) then

$$\psi = \frac{2}{N\delta_2^2} \sum_n \sum_{j=1}^{3} a_{nj} \frac{\sigma_{nj}^2}{\beta_n} \exp(\sigma_{nj}x) \sin\beta_n\xi \tag{70}$$

Setting the eigenvalue β_n equal to $2n\pi$ so that the axial velocity $\partial\psi/\partial x = 0$ at $\xi = \pm\frac{1}{2}$ gives

$$\sigma_{n1} = \frac{1}{2}(n\pi)^{1/3} \qquad \sigma_{n2} = \frac{1}{2}(n\pi)^{1/3}\left(\frac{1}{2} + \frac{3^{1/2}}{2}i\right)$$

$$\sigma_{n3} = \frac{1}{2}(n\pi)^{1/3}\left(\frac{1}{2} - \frac{3^{1/2}}{2}i\right) \tag{71}$$

This gives

$$\Gamma_1 = \sum_n \cos 2n\pi\xi \left[a_{1n} \exp(-\bar{\sigma}_n x) + a_{2n} \exp(-\bar{\sigma}_n x/2) \cos\left(\frac{3^{1/2}}{2}\bar{\sigma}_n x + \omega_n\right)\right] + C \tag{72}$$

$$\psi = \frac{1}{4R_0(Re_t r_0/l)^{1/3}} \sum_n \frac{\sin 2n\pi\xi}{2\bar{\sigma}_n} \times \left[a_{1n}\exp(-\bar{\sigma}_n x) + a_{2n}\exp(-\bar{\sigma}_n x/2) \times \cos\left(\frac{3^{1/2}}{2}\bar{\sigma}_n x + \omega_n + \frac{4}{3}\pi\right)\right] \tag{73}$$

where $\bar{\sigma}_n = (n\pi)^{1/3}/2$, and the constants a_{1n}, a_{2n}, and ω_n are to be determined by the boundary conditions on Γ, ψ, and $\partial\psi/\partial x$ at $x = 0$. Since the boundary conditions on the type-III radial boundary layer were also applied at $\eta = 1$, $x = 0$, they must also be accounted for in applying the conditions here.

The problem now requires specification of the jet distribution of the inflow through the outer cylinder to complete the solution. Without doing this, however, it is possible from Eq. (73) to see that the a_n's are of $O[R_0(Re_t r_0/l)^{1/3}]$. Thus, the perturbation circulation is of $O[R_0(Re_t r_0/l)^{1/3}]$ in this type of boundary layer. From Eq. (65) it is seen that R_0 must be less than $Re_t^{-1/2}$ to keep the linearization valid, so that the perturbation circulation is less than $Re_t^{-1/6}$.

Now the question can be answered as to what happens when the flow is introduced nonuniformly in our example of the rotating chamber. The radial velocity becomes uniformly distributed in a layer of thickness $O(Re_t^{-1/3})$ in which the change in circulation is $O[R_0(Re_t r_0/l)^{1/3}]$. Outside of this radial layer, the flow remains the same as given in Eqs. (65) through (68). This same type of adjustment layer may be expected to occur at any point in the flow where the two-dimensionality of the flow outside the end-wall boundary layers is disrupted, i.e., such a layer should occur across the length of the container if a step change in its length larger than the end-wall boundary-layer thickness occurs at some radius.

Let us return to the simple example of the porous cylindrical side walls and consider how the flow picture changes as the basic rotation is generalized from solid-body flow to a prescribed arbitrary rotation Γ_0. The nonuniform rotation of the chamber required for this is practically impossible, but this example is interesting to illustrate the manner in which the nature of the flow alters as Γ_0 = const is approached. Following the same procedure as previously, one finds that the completed flow picture within the region bounded by the radial boundary layers of $O(Re_t^{-1/4})$ on the porous cylinders at r_0 and r_i is given by

$$vr/\Omega r_0^2 = \Gamma_0 + (1 - e^{-\bar{\xi}}\cos\bar{\xi})\bar{g}R_0Re_t^{1/2} \tag{74}$$

$$\frac{\psi}{\Omega r_0^2} = -[1 - e^{-\bar{\xi}}(\cos\bar{\xi} + \sin\bar{\xi})]\,\bar{g}R_0\left(\frac{\Gamma_0\Gamma_0'}{\eta}\right)^{1/4}\frac{1}{\Gamma_0'} \tag{75}$$

$$\bar{g} = \frac{1}{N}\left(\frac{\eta}{\Gamma_0\Gamma_0'}\right)^{1/4}(2\eta\Gamma_0'' - N\Gamma_0') \tag{76}$$

$$\bar{\xi} = \left(\frac{L}{2} - |z|\right)\frac{Re_t^{1/2}}{r_0}\left(\frac{\Gamma_0\Gamma_0'}{\eta}\right)^{1/4} \tag{77}$$

Equations (74–77) reduce to Eqs. (65–68) when $\Gamma_0 = \eta$, except that Eqs. (65–68) include the flow in the radial boundary layers on the porous cylinders whereas Eqs. (74–77) do not. For a given geometry and tangential Reynolds number, the stronger the circulation gradient, the thinner the axial regions to which the flow is confined. As the circulation gradient approaches zero, the end-wall boundary layers become infinitely thick. However, the amplitude of the circulation perturbation goes to zero. For constant Γ_0, i.e., potential circulation, the stream function is completely decoupled from the circulation. The distribution of radial flow through the chamber is then the same as for the chamber at rest.

Even if the total radial flow through the chamber goes to zero, the perturbation circulation does not go to zero since the product $\bar{g}R_0$ remains finite

$$\lim_{Q\to 0} \bar{g}R_0 = \left(\frac{\eta}{\Gamma_0\Gamma_0'}\right)^{1/4} 2\eta\Gamma_0''\,\frac{L}{r_0}\frac{1}{Re_t} \tag{78}$$

This is required because each Γ_0 distribution in the flow needs a certain radial flow to sustain itself. This radial flow in the main vortex is balanced by a counter flow in the axial end-wall boundary layers, which in turn is sustained by the perturbation circulation.

VI. Conclusions

Linearizations about both simple nonrotating flows and strongly rotating flows have been considered. A weak swirl superimposed upon stagnation-point inflow has been solved in some detail. In the other extreme, after some general properties of the shear layers occurring in strongly rotating flows were considered, a weak flow through a rotating cylindrical chamber was also solved in some detail. In the swirling sink flow problem, the key assumption necessary to linearize the flow is seen to be that the swirl either is weak (large Rossby number) or is nearly independent of z. The solution illustrates the way in which an axial decay in swirl can retard the axial velocity on the axis.

The different characteristic boundary layers occurring in the linearized rotating chamber problem exhibit the way in which flows dominated by swirl are forced to be two-dimensional except for thin shear regions wherein all necessary adjustments forced by the boundary conditions are made. A large swirl in a flow tends to make the motion highly anisotropic with the resistance to radial flow much higher than to axial flow. When axial variations in tangential velocity are forced to occur, the variations take place in narrow shear layers. As a result, the fluid will gravitate to these regions of lower centrifugal force that provide a path of least radial resistance. Combined with this fact is the fact that the radial velocity supports the swirl by convection of angular momentum. Thus, the radial velocity always distributes itself axially in a way that tends to make the tangential velocity two-dimensional as far as is possible.

In strongly rotating flows, the effect of the rotation on the stream function is dependent on the gradient of the basic circulation. The larger the circulation gradient, the steeper the adjustments in the shear layers.

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Couette Flow of a Radiating and Conducting Gas

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This paper considers the Couette flow of an absorbing, emitting, and conducting gas. The combined radiation and conduction problem is treated by 1) kernel substitution and 2) radiation slip methods. Results are presented for the heat flux and temperature distributions for a gray gas. In general, there is good agreement between the kernel substitution method and the radiation slip plus conduction method for the determination of the heat flux. In the absence of conduction, the temperature distribution obtained from the kernel substitution method gives slightly better agreement with numerical results than the results obtained from radiation slip methods.

Nomenclature

k = absorption coefficient
T^* = temperature
T_0^* = reference temperature, taken to be T_W^*
T = dimensionless temperature, T^*/T_0^*
f = dimensionless freestream temperature
y = distance from left wall
I = radiation intensity
q^* = total heat flux
q = dimensionless heat flux, $q^*/\sigma T_0^{*4}$
$E_n(t)$ = exponential integral = $\int_0^1 \mu^{n-2} e^{-t/\mu} d\mu$
λ = thermal conductivity
τ^* = optical depth, $\int_0^1 k dy$
τ = $3\tau^*/2$
σ = Stefan-Boltzmann constant
μ = viscosity
u = velocity
U = velocity of upper wall
ϕ = $\mu U^2/2y_W \sigma T_0^{*4}$
ϵ = $3\lambda k/4\sigma T_0^{*3}\tau_W^2$
δ = $\epsilon \tau_W^2$
ξ = y/y_W
$\tilde{\xi}$ = $\xi/\epsilon^{1/2}$

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